In a book on convex optimization [1], exercise 2.37 page 65 states an interesting property of real-valued polynomials. I asked about how to prove this on Reddit [2] and I am going to share here their help. There is nothing new here, those are just notes. Tony Bruguier.

0.1 Statement

Let's define the polynomial p as:

$$p(t) = a_{2k}t^{2k} + a_{2k-1}t^{2k-1} + \ldots + a_1 t + a_0 \tag{1}$$

The book states that p is non-negative (i.e. $\forall t \in \mathbf{R}, p(t) \geq 0$) if and only if $p(t) = r(t)^2 + s(t)^2$ where both r and s are real-valued polynomials of degree k or less. We can assume WLOG that $a_{2k} = 1$.

In English, a real-valued non negative polynomial of degree $2k$ can be written as the sum of two squares of polynomials of degree k.

0.2 Preliminary

Here's a nifty identity:

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2 \tag{2}$$

A real-value polynomial can be factored as:

$$p(t) = \prod_j \left(t^2 + \alpha_j^2\right) \prod_j (t - \beta_j)^2 \tag{3}$$

This is because the roots are either complex, or they are double. Otherwise, the polynomial would be negative on some regions.

0.3 Proof by induction

The identity 2 can be used to prove that the statement is true for $k = 1$. let's assume that it is true for $k - 1$ and prove for k

Let p be a polynomial of degree $2k$. By equation 3, we can either pull out a factor $q(t) = \left(t^2 + \alpha^2\right)$ or $q(t) = (t - \beta)^2$, making $\frac{p}{q}$ a polynomial a degree $2k - 2$. We can thus apply the induction hypothesis and write:

$$\frac{p(t)}{q(t)} = r(t)^2 + s(t)^2 \tag{4}$$

If $q(t) = (t - \beta)^2$ we thus have:

$$p(t) = [(t - \beta)r(t)]^2 + [(t - \beta)s(t)]^2 \tag{5}$$

and the result is proven for k (you can check that the degrees work).

If $q(t) = \left(t^2 + \alpha^2\right)$ we have:

$$p(t) = (t^2 + \alpha^2)\left[r(t)^2 + s(t)^2\right] \tag{6}$$

By using the nifty identify 2 we have:

$$p(t) = [tr(t) - \alpha s(t)]^2 + [ts(t) + \alpha r(t)]^2 \tag{7}$$

and nothing is left to be proven.

0.4 Direct proof

Using the factorization 3, let's write (with the notation $i = \sqrt{-1}$):

$$A(t) = \prod_j (t + i\alpha_j) \tag{8}$$

and

$$B(t) = \prod_j (t - \beta_j) \tag{9}$$

We thus have:

$$p(t) = A(t)\overline{A(t)}B(t)^2 \tag{10}$$

Let $R(t)$ denote the real part of $A(t)$ and $S(t)$ denote the imaginary part of $A(t)$, we can rewrite 10 as:

$$p(t) = [R(t) + iS(t)]\,[R(t) - iS(t)]\,B(t)^2 \tag{11}$$

$$p(t) = [R(t)B(t) + iS(t)B(t)]\,[R(t)B(t) - iS(t)B(t)] \tag{12}$$

$$p(t) = [R(t)B(t)]^2 + [S(t)B(t)]^2 \tag{13}$$

and we are done.

References

[1] `http://www.stanford.edu/~boyd/cvxbook/` - *Convex Optimization* - Stephen Boyd and Lieven Vandenberghe

[2] `http://redd.it/ju3t5` - *Non-negative polynomials of degree 2k can be written as a sum of square of two polynomials of degree k or less* - Various Redditors